TISSERAND'S MÉCANIQUE CÉLESTE.* †

If one were asked to name the two most important works in the progress of mathematics and physics, the answer would undoubtedly be, the Principia of Newton and the Mécanique Céleste of Laplace. In their historical and philosophical aspects these works easily outrank all others, and furnish thus the standard by which all others must be measured. The distinguishing feature of the Principia is its clear and exhaustive enunciation of fundamental principles. The Mécanique Céleste, on the other hand, is conspicuous for the development of principles and for the profound generality of its methods. The Principia gives the plans and specifications of the foundation; the Mécanique Céleste affords the key to the vast and complex superstructure. It would be a mistake, of course, to suppose that Newton had no forerunners or that he had no worthy followers before Laplace. The continuity in the evolution of mechanical science is distinctly traceable, from the time of Galileo at any rate, down to the present day. But the Principia and the Mécanique Céleste present at once, more completely than any other works, the results of the great discoveries of their authors, and a perfect index to the state of contemporaneous theory. In addition, they present two distinct methods of investigation and exposition, and in this respect alone they merit the most attentive consideration.

It is natural, therefore, when a new treatise on celestial mechanics appears, to recur to the works of Newton and Laplace, to pass in review the peculiar features which render these works so specially prominent, and to align ourselves for the moment along the chain of history which connects these events with each other and with subsequent developments.

It was the happy lot of Newton to attain three brilliant achievements. First and greatest of these was the well-nigh perfect statement of the laws of dynamics; the second was the discovery of the law of gravitation; and the third was the invention of a calculus required to develop the consequences of the other two. It should be said, however, that the laws of motion were not unknown to the predecessors and contemporaries of Newton. Galileo, in fact, discovered the first two, and the third in one form or another was known to

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† Traité de Mécanique Céleste, par F. Tisserand. Paris, Gauthier-Villars et Fils. Tome I, Perturbations des planètes d'après la méthode de la variation des constantes arbitraires, 1889. Tome II, Théorie de la figure des corps célestes et de leur movement de rotation, 1891.

Hooke, Huyghens, and others; but it seems to have been the peculiar work of Newton to state these laws so clearly and fully, that the lapse of two centuries has suggested little, if any, improvement.

The law of gravitation, though commonly considered the greatest of Newton's achievements, is, in reality, less worthy of distinction than his foundation for dynamics. Its chief merit lay in the clear perception of the application of the law to the smallest particles of matter, for the mere notion of gravitation between finite masses was familiar to his contemporaries; in fact, according to Newton's own statement, the law of inverse squares as applicable to such masses was within the reach of any mathematician some years before the publication of the Principia.

The invention of the calculus, or the method of fluctions as Newton called it, was indeed a great achievement, much greater, probably, than the discovery of the law of gravitation. Unfortunately for science, however, and especially for British science, this new method of analysis figures as a silent partner in the Principia. The mathematical fashion or prejudice of Newton's day was strongly in favor of geometrical reasoning; and although the results of the Principia, as we now know, were derived by means of his calculus, he felt constrained to translate them into geometrical language. It was desirable, he thought, that the system of the heavens should be founded on good geometry. Subsequent history shows that this course was an ill-judged one. The geometrical method of the Principia renders it cumbersome, prolix, and on the whole rather repulsive to the modern reader; and the only justification which appears at all adequate for the exclusive adoption of this method, lies in the fact that his fellow countrymen would not have readily appreciated the more elegant and vastly more comprehensive analytical method. The result was very unfavorable to the growth of mathematical science in his own country. The seed he sowed took root on the continent, and has ever since grown best in French and German soil. According to Professor Glaisher, "the geometrical form of the Principia exercised a disastrous influence over mathematical studies at Cambridge University for nearly a century and a half, by giving rise to a mistaken idea of the relative power of analytical and geometrical processes."*

Readers of English mathematical text-books and treatises can hardly fail to notice the bias they show for geometrical methods, and especially for the formal, Euclidean mode of presentation, in which the procession of ideas con-

* From an address in commemoration of the bi-centenary of the publication of Newton's Principia, delivered at Cambridge University April 19, 1888. The only reference at hand to this important address is the Cambridge Chronicle and University Journal, Isle of Ely Herald, and Huntingdonshire Gazette, April 20, 1888.

sists too frequently of formidable groups of painfully accurate and technical propositions, corollaries, and scholiums. This formalism leads to a strained and unattractive literary style, which frequently degenerates into intolerable complexity or obscurity. Another and equally serious result of the apotheosis of pure geometry, is the tendency to magnify the importance of ideal problems and the work of problem solving. The exclusive pursuit of such aimless puzzles constitutes the platitude of mathematical research, though it often happens that the devotees to this species of work are mistaken for mathematicians and natural philosophers.

More than a hundred years elapsed between the publication of the Principia and the appearance of the first volumes of the Mécanique Céleste of Laplace. So slow, in fact, were mathematicians to appreciate the importance of Newton's discoveries that no improvements on his lunar and planetary theories were made before the middle of the 18th century. Continental mathematicians it would seem were obliged to develop for themselves the analytical method which Newton had used, but unwittingly discredited, before they were prepared to comprehend the scope of Newton's results. How thoroughly and completely they accomplished this work is attested by the contributions of Clairaut, Euler, D'Alembert, Legendre, Laplace, and Lagrange. The culmination of this preparatory work is exhibited more strikingly, perhaps, in the Mécanique Analytique of Lagrange, published just one hundred and one years after the Principia, than in any other single treatise. In the preface to this remarkable work Lagrange indicates in a single paragraph to what extent continental mathematicians had departed from the geometrical methods of investigation and exposition. "One will find," he says, "no diagrams in this work. The methods I expound require neither geometrical constructions nor geometrical reasoning, but only algebraical operations, subjected to a regular and uniform procedure. Those who love analysis will be pleased to see mechanics become a branch of it, and will wish me well in having thus extended its domain."

In addition to this emancipation from the narrowness of geometrical methods, continental mathematicians divested themselves from the formalism of Euclidean statement. They cultivated a literary style less distressingly precise and obtrusively technical, but immensely more luminous and attractive. The flow of ideas became with them easy and natural as well as logical; and it became less and less fashionable to inform the reader which of the ideas might be labled Proposition, Theorem, etc., or to inform him by means of the initials Q. E. D. when and where a chain of reasoning ended.

But this departure from the forms and methods of the Principia, shows

little if any detraction from the just merits of Newton's fame. Nearly every distinguished mathematician of the last generation has repeatedly acknowledged his debt and rendered his homage. They were neither blind nor servile, however, in their hero worship. They were too profoundly interested in the great problems of nature to be distracted by small issues. Animated by the zeal which such real problems impart to investigators, they sought to make their expositions comprehensive, clear, elegant, and attractive. Most of the great memoirs of this period were published in French, the language par excellence for mathematical exposition, and many of them present charms of style equal to if not surpassing those of the masters in lighter literature.

Such then was the attitude of mathematicians toward mechanical science at the end of the 18th century, when the first volumes of the Mécanique Céleste appeared. Laplace himself had done much to attain and secure this attitude, and was at this time probably the ablest investigator and expositor in the domains of mathematics and natural philosophy. The field on which he entered was a large one, and although it had been explored in many of its parts the work done was neither complete nor satisfactory. Observational astronomy and geodesy were pressing for more perfect theories and paving the way to their attainment with accumulating data. Above all, there was clearly recognized the need of a unification of the grand results which flow from the law of gravitation, and a complete exposition in one treatise of the chain of reasoning which connects the simplest concepts of matter and motion with the most complex phenomena of the material universe. This was the task which Laplace set for himself. Admirably equipped for the enterprise, and profoundly impressed with its magnitude and difficulties, he worked unceasingly for more than a quarter of a century in its accomplishment. The five volumes of the Mécanique Céleste, together with his Système du Monde, constitute the greatest systematic treatise ever published.

The treatise of Tisserand, which is the immediate object of this review, is a work in three quarto volumes, the first two of which have appeared. The first volume is devoted chiefly to the general theory of perturbations. The second volume treats of the figures of the planets and their movements of rotation. The third volume will be devoted to the theory of the moon, to an abridged theory of the satellites of Jupiter, to Hansen's method of computing perturbations, and to recent additions in celestial mechanics.

Without aiming to be thoroughly comprehensive the object of the author is elementary exposition. To those who would penetrate beyond the limits of his work, " into the more minute details of an arduous science," he tells us it goes without saying that the great treatise of Laplace will be found indispen-

sable. The author has, nevertheless, given us a very full presentation of many subjects and has devoted special attention to recent advances, so that the first two volumes bring the science substantially down to date.

We shall consider the details of the second volume only, glancing briefly at its several chapters seriatim.

The first three chapters are devoted to the general theory of attractions in conformity with the Newtonian law of inverse squares. It is worthy of remark that the potential function, which is the principal subject of exposition in these chapters, did not appear in analysis until ninety years after the publication of the Principia. Its discovery though generally credited to Laplace is due, so far as priority goes, at any rate, to Lagrange, who used it in the first instance, as others did for a long time, as a purely analytical device. The remarkable properties of this function are clearly and attractively derived by Tisserand. His account of the various theorems of Laplace, Poisson, Green, Gauss, and Chasles, cannot fail to impress the student with an appreciative sense of the profound knowledge attained by these masters, and also of the profounder ignorance which abides with us concerning the properties of matter. In reading these chapters one cannot avoid raising the query whether any additional generalizations are to be expected in this field. Those who look for such advances may find any tendency to overconfidence checked by the following brief historical summary of the principal events in the progress of the theory:

Newton, Law of Gravitation announced, ...	1687;
Lagrange, Potential Function introduced, ...	1777;
Laplace, Theorem or Equation introduced, ...	1782;
Poisson, extension of Laplace's Equation, ...	1813;
Green, Theorem and use of name Potential, ...	1828;
Gauss, General Theorems, ...	1840.

Chapter IV is devoted to the attraction of homogeneous ellipsoids, with special reference to the contributions of Lagrange, Ivory, and Gauss. Chapter V considers the attractions of homogeneous ellipsoids of revolution, together with attendant mathematical developments, and the attractions of some simple solids. Chapters VI and VII enter upon the more difficult enquiry of the equilibrium of homogeneous rotating fluid masses subject to gravitation. In these, along with applications to the sun and planets, considerable space is devoted to the Jacobian ellipsoid, which is remarkable as being a possible form of stable equilibrium, but of which no representative appears to have been discovered in nature.

At the close of the VIIth Chapter we have a most striking and suggestive theorem recently* deduced by M. Poincaré. A general criterion for the equilibrium of rotating fluid masses has long been needed, and Poincaré sought to supply this need by his theorem. It cannot be regarded in its present form as satisfactory, because it is not sufficiently precise; but it is astonishingly simple and may be susceptible of easy improvement. The result is derived by an application of Green's theorem, and as presented by Tisserand, is adapted to homogeneous masses only. But this restriction is unnecessary and does not lead to any material simplification. In its more general form the theorem may be stated thus: Let ω be the constant angular velocity and V the volume of any fluid mass M. Let u be the potential of the forces acting on a unit of this mass at any point of its surface, ∂n an element of the normal, and $d\sigma$ an element of the surface at the same point. Then, if we make one of the functions in Green's theorem equal to u and the other constant, and call f the unit of attraction, there results

$$2\omega^2 V - 4\pi f M = -\int \frac{\partial u}{\partial n}\, d\sigma.$$

Now this equation shows that in case

$$\omega^2 > 2\pi f \rho,$$

where $\rho = M/V$, or the mean density of the mass, the integral

$$\int \frac{\partial u}{\partial n}\, d\sigma$$

extended over the entire surface of M must be negative. In other words, the normal force $\partial u/\partial n$ must be directed outwards over some portions of that surface. Hence, the theorem asserts that equilibrium is impossible when the above inequality exists.

Obviously, the specification of a perfectly definite criterion is that $\partial u/\partial n$ shall be negative at *no* point of the surface. The attainment of this condition requires the removal of the integral sign in the second member of the above equation, or differentiation of the first member with respect to σ. Whether this is practicable, however, in any but simple cases is not evident.

Chapters VIII to XII treat of the equilibrium of rotating fluid masses, and especially of annular masses, subject to the attraction of adjacent bodies. They give an extended exposition of the researches of Laplace, Maxwell, Madame Kowalewski, and Poincaré concerning the form and stability of the

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* 1885, Bulletin astronomique, t. II, p. 117.

rings of Saturn. The interesting details of this subject need not be dwelt on here. We shall only quote the opinion of Tisserand relative to Maxwell's work on the rings of Saturn for the benefit of those who have found Maxwell's Electricity and Magnetism unsatisfactory reading. After explaining the general process and basis of Maxwell's investigation, Tisserand says: "The object of the memoir of Maxwell is very important, but the reasoning lacks rigor, precision, and especially clearness, and we find ourselves obliged, to our great regret, to reproduce his conclusions only."

Chapter XIII enters upon the real problem presented by the planets of a heterogeneous, rotating, spheroidal mass. It gives an admirable exposition of the theory of Clairaut founded on the recent researches of M. Hamy. Much space is devoted in Chapters XIV and XV to the investigations of Radau, Poincaré, Callandreau and others relative to the possible arrangement of density in the earth. The most important conclusion from these investigations is that if the density of the earth increases continuously from surface to centre, in whatever way, and if the surface shape of the earth is due to its original fluidity, then the surface flattening cannot exceed $\frac{1}{297.3}$.

Chapters XVI to XIX are devoted to the theory of the figures of the planets based on spherical harmonic analysis. This is the great theory of Laplace, and the chapters in question are virtually a reproduction of Laplace's work, together with an excellent exposition of the remarkable properties of his harmonic functions. The last of these chapters closes with an elegant demonstration, due to Poincaré, of the theorem of Stokes. This theorem is probably the most important addition to terrestrial mechanics made since the time of Laplace and must be considered one of the most surprising results of analysis. It amounts to saying that the potential of a planet, like the earth, rotating about a fixed axis, at any external point is determined by the shape of its sea surface, irrespective of the arrangement of the constituents of its mass. The demonstration of Poincaré is accomplished by an application of the ever-fruitful theorem of Green.

Chapters XX and XXI give a brief account of the theories of geodesy, with special reference to the determination of the dimensions and figure of the earth by means of arc measures and by means of the pendulum. In connection with the latter subject a short account is given of Helmert's condensation theory for the treatment of pendulum observations. These chapters are confessedly incomplete, but a fuller exposition was hardly desirable in view of the fact that the whole subject has been treated recently in a very thorough manner by Helmert in his admirable work entitled Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie.

Chapters XXII to XXVII are devoted to the motions of rotation of the celestial bodies and to the problems of precession and nutation which arise from the interaction of those bodies. These chapters, like Chapter XXVIII, which treats of the libration of the moon, are of interest chiefly to the specialist in physical astronomy, though well worthy of the attention of students in allied fields of research. Chapter XXIX is of more general interest, since it considers the influence of geological action on the rotation of the earth, gives an account of the researches of Hopkins, Sir William Thomson, Darwin, and others in this field, and treats at some length the question of variations of terrestrial latitudes, which is now one of special importance to astronomers and geodesists.

The last chapter treats of the rotations of bodies of variable form, of the effects of tidal friction, and of various allied questions which are not yet fully settled.

On the whole, for the ground covered by the two volumes already in print, the treatise of Tisserand appears to give the best exposition of celestial mechanics at present available. The arrangement and development of the subject are clear and orderly. The notation, which is a feature of the highest importance in such a work, conforms in general with the best modern usage. The typography has the characteristic excellence of the famous house of Gauthier-Villars et Fils, and the work has evidently been proof read with the greatest care. The specialist must, of course, consult many other treatises and memoirs, and particularly the great work of Laplace. But if to such the Mécanique Céleste is indispensable, the Traité of Tisserand can fall little short of a necessity for the reason that it records the growth of the science since the epoch of Laplace, and places the reader in line with the march of future improvements.

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